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THE WHISPERING SURFACE EFFECT†

S. V. SUKHININ

Novosibirsk (e-mail: sukhinin@hydro.nsc.ru)

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The representation theory of symmetry groups, together with variational and functional-topological methods, are used in a two-dimensional formulation to investigate the waveguide properties of one-dimensionally periodic surfaces (OPS) and interfaces. It is established that all surfaces on which the Neumann condition is satisfied possess the waveguide property—they are open waveguides. This means that there are waves localized in the neighbourhood of the surface which propagate along it without attenuation—waveguide modes. It is shown that for any hard OPS there is always a transmission band of waveguide frequencies, localized in the neighbourhood of zero—the whispering surface effect. Anomalous oscillations localized around OPSs on which the Neumann condition is satisfied are observed and investigated. Examples of surfaces for which anomalous oscillations exist and others for which none exist are presented. It is proved that OPSs on which the Dirichlet condition holds do not have a transmission band for waveguide frequencies in the neighbourhood of zero, and for some frequency bands they do not have waveguide and anomalous properties. It is shown that one-dimensionally periodic interfaces of two media possess waveguide and anomalous properties, provided that the parameters satisfy certain relationships. It is established that if the interface has the waveguide property, then transmission band of frequencies will always exist localized in the neighbourhood of zero—the whispering interface effect. An example is presented in which anomalous oscillations are investigated, dispersion relations are derived and pass and stop bands for waveguide modes are determined. © 2000 Elsevier Science Ltd. All rights reserved.

The investigation of the propagation of waves localized about one-dimensionally periodic surfaces is of interest in acoustics, water wave theory, electrodynamics, optics and other fields, since they describe the waveguide, anomalous, whispering and resonance properties of a structure. Such an investigation is difficult, however, as these properties are described by generalized eigenfunctions, and it is therefore necessary to investigate the fine structure of the continuous spectrum of the appropriate operator.

Below, continuing the investigations carried out in [1–3], we will consider waveguide, anomalous and whispering properties of one-dimensionally periodic permeable and impermeable surfaces.

1. FORMULATION AND SYMMETRY PROPERTIES OF THE PROBLEMS

Throughout this paper, the spatial variables will be non-dimensionalized relative to the minimum period of a one-dimensionally periodic surface (OPS). The problems are two-dimensional, the independent variables $(x, y) \in R^2$ are Cartesian, and the direction of spatial periodicity coincides with that of the ordinate axis. A surface or interface is described by a periodic connected curve G which divides R^2 into two connected parts, $\Omega_1 \cup \Omega_2 \cup G = R^2$ (Fig. 1). The domain $\Omega_1(\Omega_2)$ is filled by a medium all of whose parameters are labelled with subscript 1(2). Dirichlet and Neumann problems will be investigated in the domain $\Omega = \Omega_2$, in which case the subscripts will be omitted. All problems will be investigated with the oscillation domain restricted to some fundamental domain of the translation group Ω^F. No new notation will be introduced for the restrictions of Ω, Ω_1, Ω_2, G to Ω^F.

Equations and boundary conditions. If a solution of the wave equation is sought in the form $u(x, y, t) = u(x, y)\exp(-i\omega t)$, then $u(x, y)$ describes steady oscillations and

$$(\Delta + \lambda^2)u = 0 \tag{1.1}$$

where Δ is the Laplacian, $\lambda = \omega/c$ is the dimensionless oscillation frequency, c is the velocity of propagation of the waves in the oscillation domain and ω is the angular frequency of the oscillations. It should be noted that the Helmholtz equation (1.1) may be obtained from the wave equation by other means, e.g. by Laplace or Fourier transformations. The meaning of the function $u(x, y)$ and the conditions of its behaviour on G are determined by the physical content of the problem under investigation.

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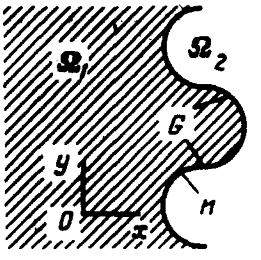

Fig. 1.

The surface G is said to be soft if

$$u(x, y)|_G = 0 \tag{1.2}$$

Problem (1.1), (1.2) will be referred to henceforth as problem D.

Let $\mathbf{n}(x, y)$ be the vector of the normal to G at a point $(x, y) \in G$. The surface G is said to be hard if

$$\partial u/\partial \mathbf{n}|_G = 0 \tag{1.3}$$

Problem (1.1), (1.3) will be referred to as problem N.

If G is the interface of the two media filling Ω_1 and Ω_2, steady oscillations in these domains are described by the relations

$$(\Delta + \lambda^2 \varkappa^2) u_1 = 0 \text{ in } \Omega_1, \quad (\Delta + \lambda^2) u_2 = 0 \text{ in } \Omega_2 \tag{1.4}$$

where $\varkappa = c_2/c_1$ is the ratio of the wave propagation velocities c_2 and c_1 in the respective media, ω is the angular frequency of the oscillations and $\lambda = \omega/c_2$ is the dimensionless oscillation frequency. The following transmission conditions must hold at the interface

$$u_1|_G = u_1|_G, \quad \partial u_1/\partial \mathbf{n}|_G = \tau \partial u_2/\partial \mathbf{n}|_G \tag{1.5}$$

The specification $\tau > 0$ depends on the physical content of the problem. For example, if one is investigating the interface properties for acoustic waves and the unknown function is the acoustic pressure, $\tau = \rho_1/\rho_2$ is the density ratio of the media.

Problem (1.4), (1.5) will henceforth be referred to as problem T. For convenience it will be assumed that $\varkappa > 1$ and $\tau \ll 1$. This is true, for example, if G is an interface between air and water (a water wave).

Symmetry properties. Since the Laplacian is invariant with respect to any locally-plane symmetries, the symmetry of problems D, N and T is determined by the shape of the surface G. All the OPSs can be classified by their groups of admissible symmetries. Only two types of such surfaces are possible [4, 5]. The first type is a surface that admits of only the translation group $\{T_1\}$, $T_1\langle(x, y)\rangle = (x, y + 1)$. The second type is that admitting of the group $\{T_1, D^x{}_1\}$; $D_1^x\langle(x, y)\rangle = (x, -y)$—mirror reflection in the abscissa axis. No OPSs exist that admit of other symmetry groups [4, 5]. An example of a surface of the second type is shown in Fig. 1.

Transformations in the symmetry group S_G of the surface G map a solution of problem D, N or T into a solution of the same problem. Therefore, the space of admissible solutions of any of these problems is the sum of invariant subspaces with respect to an irreducible representation of S_G in the solution space. We may thus replace the space of admissible solutions of problem D, N or T by one of these subspaces, simplifying the investigation.

Our most important task is to investigate boundary-value problems in solution spaces which are invariant with respect to irreducible representations of the group $\{T_1\}$. The space of admissible solutions of any homogeneous problem for the wave equation with boundary conditions on an OPS is the sum of subspaces which are invariant with respect to an irreducible representation of the group $\{T_1\}$. If a function $u(x, y)$ belongs to such a subspace, then for some ξ, $-\pi \leqslant \xi \leqslant \pi$

$$T_1\langle u(x, y)\rangle = u(x, y+1) = e^{i\xi} u(x, y) \tag{1.6}$$

The quantity ξ describes the phase shift of the oscillations in adjoining fundamental domains of the translation group. In what follows, unless otherwise stated, it is assumed that $0 < \xi \leqslant \pi$.

Condition (1.6) means that oscillations in adjoining fundamental domains of the translation group occur with a phase shift ξ. The general solution of problem D, N or T is a superposition of solutions of type (1.6) with respect to ξ.

The following proposition may be verified by a direct check.

Lemma 1.1. If a function $u(x, y)$ satisfies condition (1.6), then in free space

$$u(x,y) \equiv e^{i\xi y} v(x,y), \quad v(x,y+1) \equiv v(x,y) \tag{1.7}$$

Identity (1.7) is sometimes referred to as Floquet's theorem, or as Rayleigh–Bloch waves, and (1.6) as the phase shift condition for oscillations in adjoining fundamental domains of the translation group.

In the case of a surface G of the second type, there are four possible one-dimensional irreducible representations of the group $S_G = \{T_1, D_1^x\}$ in the space of admissible solutions of problem D, N or T [6]

$$\begin{gathered}\{\tau_1(T_1) = -1, \tau_1(D_1^x) = +1\}, \ \{\tau_2(T_1) = -1, \tau_2(D_1^x) = -1\} \\ \{\tau_3(T_1) = +1, \tau_3(D_1^x) = +1\}, \ \{\tau_4(T_1) = +1, \tau_4(D_1^x) = -1\}\end{gathered} \tag{1.8}$$

$\tau_k (k = 1, \ldots, 4)$ are irreducible representations of the group $\{T_1, D_1^x\}$.

Problems D, N and T with the additional condition (1.7) are known as problems $D(\xi)$, $N(\xi)$ and $T(\xi)$. The self-adjoint extensions of the Laplacian for $D(\xi)$ and $N(\xi)$ will be denoted by $-\Delta^{\xi}{}_D$ and $-\Delta^{\xi}{}_N$, respectively. We will consider self-adjoint extensions of the Laplacian in the space $L_2(\Omega)$ and the corresponding self-adjoint restrictions of this extension [7].

Waveguide, anomalous and whispering properties. At the physical level of rigor, a surface possesses the waveguide property if travelling waves exist localized in its neighbourhood that propagate along it without attenuation (waveguide modes); it has the anomalous property if waves exist localized in its neighbourhood which are in phase in all fundamental domains of the translation group; and it has the whispering property if a transmission band exists for waveguide mode frequencies in the neighbourhood of zero. For the subsequent discussion we have to formalize the terminology.

Definition 1.1. A waveguide function of problem D, N or T is a generalized eigenfunction of problem $D(\xi)$, $N(\xi)$ or $T(\xi)$, $0 < |\xi| \leqslant \pi$ localized in the neighbourhood of an OPS. The corresponding oscillation frequency is called a waveguide frequency. A hard or soft OPS or interface has the waveguide property if a non-trivial waveguide function exists corresponding to problem D, N or T.

The general solution of equations (1.1) or (1.4) with condition (1.6) in $\Omega_k (k = 1, 2)$ for $|x| \gg 1$ has the form

$$u_k(x,y) = \sum_{n=-\infty}^{+\infty} \exp[i(2\pi n + \xi)y][b^{(+)}_{(n,k)} \exp(|x|\beta_{(n,k)}) + b^{(-)}_{(n,k)} \exp(-|x|\beta_{(n,k)})], \quad k = 1,2$$

$$\beta_{(n,1)} = \sqrt{(2\pi n + \xi)^2 - \lambda^2 \varkappa^2}, \quad \beta_{(n,2)} = \sqrt{(2\pi n + \xi)^2 - \lambda^2}$$

Therefore, a function $u^*(x, y)$ is localized about G if and only if

$$u_k^*(x,y) = \sum_{n=-\infty}^{+\infty} b^{(\pm)}_{(n,k)} \exp[i(2\pi n + \xi)y] \exp(-|x|\beta_{(n,k)}), \quad k = 1,2 \tag{1.9}$$

and $\mathrm{Re}(\beta_{(n,k)}) > 0$ $(k = 1, 2)$ for all non-trivial terms.

Remark 1.1. Suppose that for some λ_*, $0 < \lambda_* < \xi$, an eigenfunction of problem $D(\xi)$, $N(\xi)$ or $T(\xi)$ exists which, for $|x| \gg 1$, has the form (1.9), $\mathrm{Re}(\beta_{(n,k)}) > 0$ $(k = 1, 2)$; then this eigenfunction has the waveguide property. A representation of the solution of these problems for $|x| \gg 1$ in the form (1.9) is called radiation conditions.

Remark 1.2. Since a waveguide function $u^*(x, y)$ in free space admits of a representation $u_*(x, y) \equiv \upsilon(x, y) \exp(i\xi y)$, it follows that ξ may be regarded as the wave number and the function $\upsilon(x, y) \equiv \upsilon(x, y + 1)$ as the amplitude. Elementary waveguide packets have the form

$$u_*(x, y, t) = v(x, y) \exp [i(\xi y - \omega_* t)]$$

where ω_* is the corresponding dimensional waveguide frequency and ξ varies in the half-open intervals $\Xi^{(-)} = \{\xi: -\pi \leq \xi < 0\}$, $\Xi^{(+)} = \{\xi: 0 < \xi \leq \pi\}$. The propagation direction of the wave packets is determined by the sign of ξ.

Sets of waveguide frequencies $\sigma_{(k)} = \{\lambda^{(k)}(\xi), \xi \in \Xi^{(-)} \cup \Xi^{(+)}\}$ $(k = 1, \ldots, K)$ which are connected in the topology of the real axis will henceforth be called transmission bands: to any transmission band there corresponds a definite waveguide mode.

Definition 1.2. An OPS possesses the whispering property if a transmission band σ_1 exists such that $\sigma_1 = (0, \lambda^*_1]$, $\lambda^*_1 \neq 0$.

Solutions of problems $D(0) = D(\xi)|_{\xi=0}$, $N(0) = N(\xi)|_{\xi=0}$ and $T(0) = T(\xi)|_{\xi=0}$ satisfy the condition

$$u(x, y + 1) = u(x, y) \tag{1.10}$$

which corresponds to zero phase shift of the oscillations or describes in-phase oscillations in adjoining fundamental domains of the translation group.

Definition 1.3. Generalized eigenfunctions (eigenfrequencies) of problems $D(0)$, $N(0)$ or $T(0)$ localized in a neighbourhood of G will henceforth be called anomalous functions (frequencies) of those problems. Anomalous functions describe in-phase oscillations localized around a hard or soft OPS or interface. A one-dimensionally periodic structure possesses the anomalous property if a non-trivial anomalous function of the corresponding problem exists.

2. A SOFT SURFACE

Wave propagation about a soft surface is described by a Dirichlet problem for the wave equation. For example, in problems of mechanics a surface is soft if it is free. Characteristic problems involve the propagation of longitudinal acoustic waves in an elastic body about the boundaries of the body and the propagation of acoustic waves in water about the interface with air. In this section it will be proved that a soft surface never has the whispering property, a stop band will be indicated for low frequencies, and conditions will be presented under which the surface has neither waveguide nor anomalous properties.

The well posedness of the problem. To establish that the family of problems $D(\xi)$ is well posed one must show that the corresponding inhomogeneous problems are solvable for at least some values of the dimensionless frequency. The following proposition may be verified by direct calculation.

Lemma 2.1. Green's function for Eq. (1.1) with conditions (1.6) has the form

$$E(x - x_0, y - y_0, \lambda) = \sum_{n=-\infty}^{n=\infty} \frac{\exp\left[i(2\pi n + \xi)(y - y_0) - |x - x_0| \sqrt{(2\pi n + \xi)^2 - \lambda^2}\right]}{-2\sqrt{(2\pi n + \xi)^2 - \lambda^2}} \tag{2.1}$$

The choice of a fundamental solution of problem $D(\xi)$ in the form (2.1) enables potential theory to be used to prove that the problem is well posed. Existence theorems have been proved for solutions of the inhomogeneous problem $D(\xi)$, provided the dimensionless frequency is such that the solution of the homogeneous problem is unique [8].

Modification of classical relations. To investigate waveguide, anomalous and whispering properties of a soft OPS, we need a modification of certain classical identities and inequalities for functions that satisfy the phase shift condition for oscillations in adjoining fundamental domains of the translation group. Earlier results [9, 10] will be used in part.

Rellich's inequality. Below we will use the method of [11], modified in [12] for functions satisfying the phase shift condition for oscillations in adjoining fundamental domains of the translation group (1.6). Let $u(x, y)$ be some solution of problem $D(\xi)$, and let $\mathbf{n}$ be a unit vector along the normal pointing from the oscillation domain to the surface G. We have an equality

$$\int_G (x+b)(\mathbf{n},\mathbf{e}_x)\left|\frac{\partial u}{\partial \mathbf{n}}\right|^2 dG = 2\iint_\Omega \left|\frac{\partial u}{\partial x}\right|^2 d\Omega \tag{2.2}$$

where $\mathbf{e}_x$ is the unit vector of the abscissa axis, and $(\mathbf{n}, \mathbf{e}_x)$ denotes the scalar product.

This equality (Rellich's equality [11] holds for all solutions of problem $D(\xi)$ that attenuate at infinity along the abscissa axis.

Friedrichs' inequality. Let $g(x,y)$ be a sufficiently smooth, bounded, real-valued function such that $g(x, y+1) \equiv g(x,y)$ and $g(x,y) \neq 0$, $(x,y) \in \Omega$. Any function $u(x,y)$ satisfying condition (1.6) may be represented as $u(x,y) = g(x,y)\,\upsilon(x,y)$. The following inequality [12], usually known as Friedrichs' inequality [9], holds for any function $u(x,y)$ that decreases as the distance from a soft boundary increases

$$\int_G |\upsilon|^2 g \frac{\partial g}{\partial \mathbf{n}} dG - \iint_\Omega (|\upsilon|^2 g\Delta g) d\Omega \leq \iint_\Omega |\nabla u|^2 d\Omega \tag{2.3}$$

Non-existence of waveguide and anomalous properties of a soft surface.

Definition 2.1. An OPS is said to be normally illuminated if $(\mathbf{n}, \mathbf{e}_x) \leq 0$.

An example of a normally illuminated periodic surface is given in Fig. 1. The reason for the name is that the surface is completely illuminated by parallel rays incident upon it along the normal to the direction of periodicity.

Theorem 2.1. If an OPS is normally illuminated, then it possesses neither the waveguide nor the anomalous property.

Proof. Let $u(x,y)$ be a waveguide or anomalous solution of the homogeneous boundary-value problem D. Since the surface is normally illuminated, a b exists such that $(x+b)\,(\mathbf{n}, \mathbf{e}_x) \leq 0$ for all points of G. By Rellich's equality (2.2), $\partial u/\partial x \equiv 0$. Since $u(x,y)$ decreases as the distance from G increases, it follows that $u(x,y) \equiv 0$ in Ω.

Theorem 2.2. Soft surfaces do not possess the whispering property. No waveguide frequencies of a soft OPS exist in the half-open interval $[0, \pi)$.

Proof. Problem $D(\xi)$ with additional homogeneous Neumann conditions on a set of straight lines $R = \{x = R_m, m = 1, 2, \ldots, M\}$ will be denoted by $D(N, R)$. Let λ_{NR} be either the least eigenvalue of that problem or the lowest frequency of the continuous spectrum and let $\lambda(\xi)$ a waveguide or anomalous frequency of problem $D(\xi)$. For all R

$$\lambda_{NR} \leq \lambda(\xi) \tag{2.4}$$

If R values exist for which $0 < \xi \leq \lambda_{NR}$, it follows from (2.4) that no waveguide or anomalous frequencies of problem $D(\xi)$ exist in the interval $(0, \xi)$.

Let

$$x_* = \min_{(x,y)\in G}(x), \quad x^* = \min_{(x,y)\in G}(x)$$

and let the points (x_*, y_*), (x^*, y^*) divide the surface G into connected components (Fig. 2). We may assume that $x_* = 0$ and that y^* are integers. Let g be some connected component of this partition. The domain $\Omega \cap \{(x,y)\colon x_* < x, x^*\}$ can be covered by a finite set of rectangles with sides parallel to the coordinate axes (Fig. 2), of width $1+\varepsilon$, $\varepsilon > 0$, in the direction of the ordinate axis. We will refer to this cover henceforth as an ε-cover. Let $\{R_m, m = 1, 2, \ldots, M\}$ be the coordinates of the vertices of the rectangles along the abscissa axis, let P be some rectangle, and let λ_{NP} be an eigenfrequency of problem D restricted to P with homogeneous Neumann conditions on the lateral sides of P. Taking $g = \sin[\pi(y - y_P)/(1+\varepsilon)]$ in Friedrich's inequality (2.3), where y_P is the ordinate of the lower side of P, we obtain

$$\left(\frac{\pi}{1+\varepsilon}\right)^2 \leq \left(\iint_{\Omega\cap P} |\nabla u|^2 d\Omega\right)\left(\iint_{\Omega\cap P} |u|^2 d\Omega\right)^{-1}$$

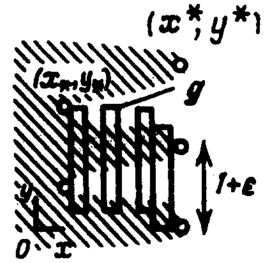

Fig. 2.

Since P is an arbitrary rectangle in the ε-cover, inequality (2.4) becomes $\pi/(1 + \varepsilon) \leqslant \lambda_{NR} \leqslant \lambda(\xi)$. Since ε is an arbitrary number, this implies the assertion of the theorem.

Remark 2.1. We know of no examples of soft OPSs possessing waveguide or anomalous properties. It may be that no such surfaces exist.

3. A HARD SURFACE

Typical examples of problems involving a hard OPS are the propagation of acoustic waves over impermeable surfaces and the propagation of water waves along the shore line. It can be proved, using representation theory for symmetry groups and variational methods, that any hard surface possesses the waveguide property and there is always a transmission band in the low frequency domain. The physical meaning of this statement is that for acoustic waves there is always a whispering surface effect, and for water waves one-dimensionally periodic shore lines are tsunami waveguides. Since tsunamis are long waves, they are described even in the linear theory by problem N [13].

Variational formulation. Identifying points $(x, y) \equiv (x, y + 1)$ throughout the oscillation domain Ω, we obtain a corresponding domain Ω^z on the surface of a cylinder of diameter 1. If $H^1(\Omega^z)$ is the Sobolev space for functions defined on Ω_z, then, by condition (1.6) and Lemma 1.1, the space of admissible solutions $H^1_\xi(\Omega)$ of problem $N(\xi)$ has the form

$$H^1_\xi(\Omega) \equiv \exp(i\xi y)H^1(\Omega^z)$$

Various known methods [7, 9, 11, 14, 15] may be used for the variational formulation of problem $N(\xi)$ in the space of admissible solutions $H^1_\xi(\Omega)$.

Existence of the waveguide property. Solutions of problem $N(\xi)$ in the space $H^1_\xi(\Omega)$ that describe waveguide or anomalous properties are eigenfunctions of the self-adjoint extension—Δ^ξ_N of the Laplacian, and the corresponding eigenvalues belong to the pure point spectrum of $-\Delta^\xi_N$. A special feature in the investigation of the waveguide properties of an OPS is that the operator $-\Delta^\xi{}_N$ has a continuous spectrum $\Sigma^\xi_N = [\xi^2, \infty)$. The existence of waveguide and anomalous frequencies has been investigated using the method known as "Dirichlet-Neumann bracketing" [9, 11, 14, 15]. Suppose that either Dirichlet or

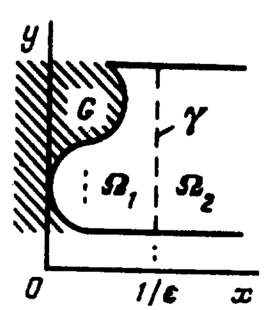

Fig. 3.

Neumann conditions hold on $\gamma = \{(x,y): x = 1/\varepsilon\}$, $0 < \varepsilon < \infty$ (Fig. 3). In what follows we will denote problem $N(\xi)$ with a Dirichlet condition by $N(\xi, \varepsilon, D)$ and with a Neumann condition by $N(\xi, \varepsilon, N)$. Let $\lambda^{(k)}_{D\varepsilon}$ be the eigenfrequencies of problem $N(\xi, \varepsilon, D)$, let $\lambda^{(k)}_{N\varepsilon}$ be those of problem $N(\xi, \varepsilon, N)$ and let $\lambda^{(k)}(\xi)$ be an eigenfrequency of problem $N(\xi)$, with frequencies numbered in increasing order ($k = 1, 2, \ldots, K$). Since the Neumann condition enlarges the space of solutions while the Dirichlet condition reduces it, it follows that for all ε

$$0 \leqslant \lambda^{(k)}_{N\varepsilon} < \lambda^{(k)}(\xi) < \lambda^{(k)}_{D\varepsilon}, \quad k = 1, \ldots, K \tag{3.1}$$

Remark 3.1. If ε and K_* exist for which $0 < \lambda^{(K^*)}_{N\varepsilon}$ and $\lambda^{(K^*)}_{D\varepsilon} < \xi$, then (3.1) implies the existence of at least K_* waveguide frequencies of problem $N(\xi)$ for waves travelling in one direction along the surface.

Theorem 3.1. A hard OPS possesses the waveguide and whispering properties.

Proof. As shown in Fig. 3, let

$$x_* = \min_{(x,y)\in G}(x) = 0, \quad x^* = \max_{(x,y)\in G}(x) \tag{3.2}$$

The eigenfrequencies of problem $N(\xi)$ are bounded away from zero. Suppose, on the contrary, that $\lambda^{(1)}_{N\varepsilon} = 0$ and $u^{(1)}_{N\varepsilon} = \text{const}$. If $G \subset \{(x,y): |x| < 1/\varepsilon$ and $\varepsilon < 1/x^*$, then the fact that the domain Ω_1 is connected (Fig. 3) and condition (1.6) imply that $u^{(1)}_{N\varepsilon} \neq$ const, whence we obtain $u^{(1)}_{N\varepsilon} \neq$ const; therefore, $0 < \lambda^{(1)}_{N\varepsilon}$.

An upper bound for problem $N(\xi)$ is obtained by constructing trial functions for the variational formulation. Let

$$\Omega_1 = \Omega \cap \{(x,y): x_* < x < x^*\}, \quad \Omega_2 = \Omega \cap \{(x,y): x^* < x\}$$

Any solution $u(x, y)$ of problem $N(\xi)$ may be expressed as the sum of a discontinuous function u_d and a continuous function u_c

$$u = u_d + u_c; \quad u_d = u_d(x,y) \in \hat{H}^1(\Omega_1), \quad u_c = u_c(x,y) \in H^1_\xi(\Omega_F) \tag{3.3}$$

where $\hat{H}^1(\Omega_1)$ is the Sobolev space of functions that vanish on γ and Ω_F is a fundamental domain of the translation group in R^2. The function $u_d(x, y)$ may be completed over the whole oscillation domain by using condition (1.6).

Let $u_c(x,y) = \exp(i\xi y)\cos(\pi x\varepsilon/2)$. This function satisfies all relationships of problem $N(\xi, \varepsilon, D)$ except for the impermeability condition, and it is continuous throughout the space. The function u_d may be expressed as

$$u_d = \begin{cases} \varkappa \cos\dfrac{\pi x}{2|x^*|}, & x_* < x < x^*, \ 0 < y < 1 \\ 0, & |x| > x^*, \ 0 < y < 1 \end{cases} \tag{3.4}$$

($\varkappa$ is a parameter). The following relationship reflects the variational property of the eigenvalues

$$(\lambda_{D\varepsilon})^2 \leqslant \frac{\int |\nabla u|^2 \, d\Omega_\varepsilon}{\int |u|^2 \, d\Omega_\varepsilon} =$$

$$= \frac{\int \left[|\nabla u_c|^2 + (\nabla \bar{u}_c)(\nabla u_d) + (\nabla u_c)(\nabla \bar{u}_d)^* + |\nabla u_d|^2\right] d\Omega_\varepsilon}{\int \left[|u_c|^2 + \bar{u}_c u_d + \bar{u}_d u_c + |u_d|^2\right] d\Omega_\varepsilon} = \mu^2(\varkappa, \varepsilon) \tag{3.5}$$

where integration is performed over the domain $\Omega_\varepsilon = \Omega \cap \{(x,y): |x| < 1/\varepsilon\}$, and a bar over a symbol denotes its complex conjugate. For small ε

$$\mu^2(\varkappa, \varepsilon) = \xi^2 + \varepsilon(a\varkappa + b\varkappa^2) + O(\varepsilon^2)$$

For small ε and $\varkappa$, the number

$$a = \lim_{\varepsilon\to\infty} \frac{\int |\partial u_c/\partial y|^2 \, d\Omega_\varepsilon}{\varepsilon\int(|u_c|^2 + u_c^* u_d + u_d^* u_c)d\Omega_\varepsilon} \neq 0$$

is well defined. Consequently, for small (positive or negative) $\varkappa$, we have $a < 0$, and therefore $\mu^2(\varkappa, \varepsilon) < \xi^2$.

The following proposition may be proved using methods of the theory of self-adjoint operators based on the eigenvalue theorem [12, 15].

Theorem 3.2. The spectrum of the operator—$\Delta^{\xi}{}_N$ is discrete below ξ^2 and contains at most K^* and at least K_* pure point eigenvalues. The numbers K^* and K_* are defined by the inequalities

$$\lambda_{N\varepsilon}^{(K^*)} < |\xi| \leq \lambda_{N\varepsilon}^{(K^*+1)}, \; \lambda_{D\varepsilon}^{(K_*)} < |\xi| \leq \lambda_{D\varepsilon}^{(K_*+1)}$$

Definition 3.1. A hard OPS of the form shown in Fig. 4 is known as a simple (L–d)-comb. If $d = 0$, the surface is called a simple L-comb.

Using inequalities (3.1) and Theorem 2.2, one can estimate the number of waveguide modes of a hard surface in the intervals $\xi \in \Xi^{(-)}$ or $\xi \in \Xi^{(+)}$. If the surface is a simple (L–d)-comb, then by Theorem 3.1 there will always be at least one waveguide mode.

Theorem 3.3. Every simple (L–d)-comb, $L > 1/2$, has at least $K_{\min}$ waveguide modes in the interval $\Xi^{(+)}$ or $\Xi^{(-)}$, $K_{\min} < L + 1/2 \leq K_{\min} + 1$. If $L + 1/2$ is not an integer, than $K_{\min}$ is its integral part: $K_{\min} = [L + 1/2]$. If $L + 1/2$ is an integer, then $K_{\min} = [L - 1/2]$.

Remark 3.2 (The whispering gallery effect). Since the propagation of acoustic waves over hard surfaces is described by boundary-value problems for the wave equation with Neumann boundary conditions, the family of problems $N(\xi)$ describes the propagation of elementary acoustic waveguide packets with wavenumber ξ localized about the surface. By virtue of the propositions we have proved, for any hard surface a transmission band exists localized in the neighbourhood of zero (low frequencies). This means that all OPSs possess the whispering property. Possibly, this property of hard surfaces explains the "whispering gallery" effect.

Remark 3.3 (Tsunami waveguides). Since long waves in shallow water about a shore line are described by problem N, our theorems yield a solution of the well-known Lavrent'yev problem on tsunami waveguides [13]. In view of the existence of a transmission band for any one-dimensionally periodic coast line in the low-frequency band and the fact that a tsunami is a packet of long waves, we can state that any one-dimensionally periodic coast line is a tsunami waveguide.

The anomalous property. If the surface admits of the symmetry group $\{T_1, D_1^x\}$, then four irreducible one-dimensional representations (1.8) of this group exist in the solution space. The representations τ_1 and τ_2 conform to the conditions of Theorem 2.1 for $\xi = \pi$; for τ_3 and τ_4 the conditions of that theorem are unsuitable, since $\xi = 0$. For the representation τ_4 of (1.8) we have the identities

$$u(x, y + 1) \equiv u(x, y), \; u(x, -y) \equiv -u(x, y) \tag{3.6}$$

It can be verified by direct calculation that the space of admissible solutions of problem $N(0)$ in the class of functions satisfying these identities is a closed subspace of $H^1_\xi(\Omega)$; the continuous spectrum of the operator $-\Delta^0{}_N$ in that space has the form $\Sigma_N = [4\pi^2, +\infty)$.

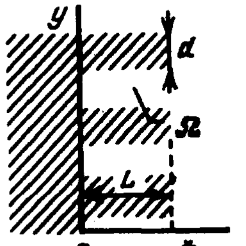

Fig. 4.

In what follows we use the notation and method of Theorem 2.1.

Theorem 3.4. A simple $(L\text{–}d)$-comb possesses the anomalous property if $0 \leq d < 1/2$.

Proof. The theorem is true if $\lambda_{D\varepsilon} < 2\pi$ exists for some ε, $\lambda_{D\varepsilon}$ being an eigenfrequency of problem $N(0, \varepsilon, D)$ with conditions (3.6).

The lower bound is proved indirectly: if $\lambda^{(1)}_{N\varepsilon} = 0$, then $u^{(1)}_{N\varepsilon} \equiv 0$ by (3.6).

The upper bound is proved by using the variational property of eigenfrequencies. Putting

$$u_c(x, y) = \sin(2\pi y)\cos(\pi\varepsilon x/2),\ u_d(x, y) \equiv 0$$

in (3.5), we find that for small ε

$$\mu^2(\varkappa,\varepsilon) = 4\pi^2 + A\varepsilon + O(\varepsilon^2),\ \ A = 16L\pi\cos(\pi d/2)\sin(\pi d/2)[1 - 2\cos^2(\pi d/2)] < 0$$

Theorem 3.5. A hard OPS having the form

$$G = \left\{(x,y): y = \frac{L-x}{2L},\ \ 0 \leq x \leq L, 0 \leq y \leq \frac{1}{2}\right\} \cup$$

$$\cup \left\{(x,y): y = \frac{L+x}{2L},\ \ 0 \leq x \leq L, \frac{1}{2} \leq y \leq 1\right\}$$

in a fundamental domain of the translation group (see Fig. 5) always possess the anomalous property. Such an OPS is usually known as an echelette.

Proof. Suppose that in (3.3)

$$u_c(x,y) = \sin(2\pi y)\cos(\pi x\varepsilon/2)$$

$$u_d = \begin{cases}(y - ½)\varkappa\cos[\pi x/(2L)], & 0 < x < L, 0, < y < 1\\ 0, \quad x > L, & 0 < y < 1\end{cases}$$

The functions $u = u_d + u_c$ satisfies the conditions of problem $N(0, \varepsilon, D)$ and (3.6). For small ε, we have

$$\mu^2(\varkappa, \varepsilon) = 4\pi^2 + \varepsilon\,(a\varkappa + b\varkappa^2)\ \ + O\,(\varepsilon^2) \tag{3.7}$$

Since $a = 32L\,(10 - 3\pi)$, this implies, by (3.5), that for small negative $\varkappa$

$$\mu^2(\varkappa, \varepsilon) < 4\pi^2 \tag{3.8}$$

Theorem 3.6. A hard surface of the form $G = \{(x,y): x = L[1 + \cos(2\pi y]/2\}$ possesses the anomalous property for all $L > 0$.

Proof. The functions in representation (3.3) may be chosen as

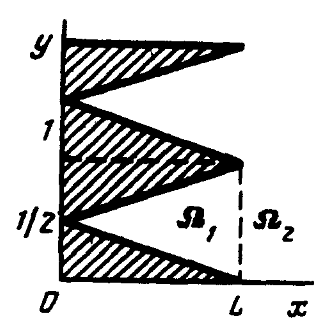

Fig. 5.

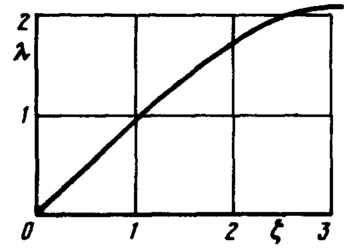

Fig. 7.

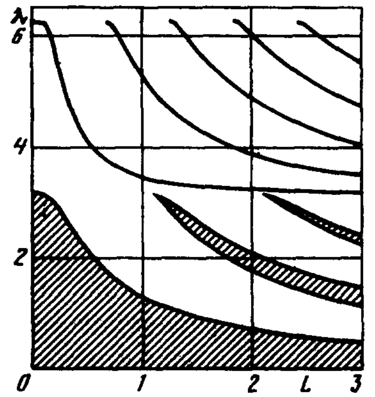

Fig. 8.

Figure 8 shows the width of the transmission band and the anomalous frequencies as functions of the length of the "teeth" of a simple L-comb. The passbands are shown hatched; the solid curves are graphs of the anomolous oscillations as a function of the length of teeth of the simple L-comb. A numerical investigation of the passband as a function of the geometric parameter of a simple comb was carried out with the help of the dispersion relations (3.9). The dependence of anomalous frequencies of a simple L-comb on the geometric parameter L have been investigated by direct allowance for the finiteness of the energy [16].

4. A PERMEABLE SURFACE

Problem T is not a diffraction problem [17, 18]. If the parameter τ occurring in (1.4) tends to zero, then problem T splits into a problem N in the domain Ω_1 and a problem D in the domain Ω_2, both of which have just been investigated. In this section we will use functional-topological methods to show that, at small (large) values of $\tau > 0$, the waveguide, anomalous and whispering properties of an interface differ only slightly from the same properties for the corresponding limiting cases of a surface which is soft on one side and hard on the other.

Operator formulation of the problem. Using Green's function (2.1), we can formulate problem $T(\xi)$ in terms of operators. Let

$$E_1(x,x_0,y,y_0,\lambda)=E(x-x_0,y-y_0,\lambda\varkappa),\quad E_2(x,x_0,y,y_0,\lambda)=E(x-x_0,y-y_0,\lambda)$$

If the solution of problem $T(\xi)$ in the domains Ω_1 and Ω_2 is sought in the form of single- and double-layer potentials

$$u_k(x,y)=\int_G\left[E_k(x,y,x_0,y_0)\mu(x_0,y_0)+\frac{\partial E_k(x,y,x_0,y_0)}{\partial\mathbf{n}(x_0,y_0)}[\tau(2-k)+(k-1)]\nu(x_0,y_0)\right]dG \tag{4.1}$$

$k=1,2$

then conditions (1.5) at the interface take the form presented in [19], r and s are the coordinates of the natural parametrization of the interface and $\mu(s)$ and $\nu(s)$ are densities of the single- and double-layer potentials

$$\mu(r)=\frac{2}{1+\tau}\frac{\partial}{\partial\mathbf{n}(r)}\int_G\left[\left(\frac{\partial E_2(r,s)}{\partial\mathbf{n}(s)}-\frac{\partial E_2(r,s)}{\partial\mathbf{n}(s)}\right)\tau\nu(s)+(\tau E_2(r,s)-E_1(r,s))\mu(s)\right]ds$$
$$\nu(r)=-\frac{2}{1+\tau}\int_G\left\{\left(\frac{\partial E_2(r,s)}{\partial\mathbf{n}(s)}-\tau\frac{\partial E_1(r,s)}{\partial\mathbf{n}(s)}\right)\nu(s)+[E_2(r,s)-E_1(r,s)]\mu(s)\right\}ds \tag{4.2}$$

Let $H = L_2(G) \times L_2(G)$. It can be verified directly that the operator $F(\lambda, \varkappa, \xi, \tau) : H \to H$ defined by the right-hand sides of (4.2) is compact.

The Riemann surface. The operator $F(\lambda, \kappa, \xi, \tau)$ as a function of λ is conveniently investigated on the Riemann surface of its analytic continuation as a function of λ. Suppose the numbers $\{|2\pi k \pm \xi|,\ (2\pi k \pm \xi)/\varkappa\ |\}_{k=0,1,2...} = \{\beta_k\}_{k=1,2...}$ are indexed in increasing order, and that $f(\lambda) = h_1 * F(\lambda, \varkappa, \xi, \tau)\langle h_2\rangle$, where h_1 and h_2 are certain elements of H (the asterisk denotes the scalar product in H). The construction and properties of the Riemann surface $\Lambda(\xi)$ of the function $f(\lambda)$ were described in [1, 2].

Remark 4.1. The Riemann surface $\Lambda(\xi)$ is infinite-sheeted: the points $\pm\beta_k$ $(k = 1, 2, \ldots)$ are second-order branch points; for each $\lambda \in \Lambda(\xi)$ a natural number $K(\lambda)$ exists such that for $k > K(\lambda)$ we have $\mathrm{Re}(\sqrt{(\beta_k^2 - \lambda^2)}) > 0$.

Definition 4.1. Suppose that for some $\lambda^* = \lambda^*(\xi) \in \Lambda(\xi)$ a non-trivial solution of problem $T(\xi)$ in the form (4.1) exists. Then λ^* is called a quasi-eigenfrequency, and the corresponding solution u^* is called a quasi-eigenfunction of the problem.

The direction of propagation along the abscissa axis for each propagating mode is determined by the signs of $\mathrm{Im}\,(\sqrt{(\beta_k^2 - \lambda^2)}) = \mathrm{Im}(\beta_{(k,2)})$ and $\mathrm{Im}\,(\sqrt{(\beta_k^2 - \varkappa^2\lambda^2)}) = \mathrm{Im}(\beta_{(k,1)})$. In the general case, quasi-eigenfunctions may contain modes propagating in different directions along the abscissa axis. Such quasi-eigenfunctions are of interest in themselves.

Definition 4.2. The quasi-eigenfunctions u^* (quasi-eigenfunctions λ^*) of problem $T(\xi)$ are called eigenfunctions (eigenfrequencies) of the problem if u^* for $|x| \gg 1$ has the form (1.9) with $\mathrm{sign}\,[\mathrm{Im}(\beta_{(n,1)})] = \mathrm{sign}\,[\mathrm{Im}(\beta_{(n,2)})] = \mathrm{const}$ for all integers n.

The physical meaning of quasi-eigenfunctions and quasi-eigenfrequencies depends on the content of the problem.

The following proposition is proved indirectly [2, 3].

Lemma 4.1. The eigenfrequencies of problem $T(\xi)$ can only be real.

The analytical Fredholm theorem. To prove that problem $T(\xi)$ is well posed, we must prove that it has a unique solution for at least some values of $\lambda \in \Lambda(\xi)$. Let $\Lambda_0(\xi) = \{\lambda\colon \mathrm{Re}\sqrt{(\beta_n^2 - \lambda^2)} > 0, n = 1, 2, \ldots\}$ be a sheet of the Riemann surface $\Lambda(\xi)$. The following lemma is true [3].

Lemma 4.2. If the inhomogeneous problem $T(\xi)$ for $\lambda \in \Lambda_0$, $\mathrm{Im}\,(\lambda) \neq 0$ has a solution, then it is unique.

Lemma 4.2 and Theorem 3.41 of [19] imply the following

Theorem 4.1. If $\lambda \in \Lambda_0(\xi)$, the inhomogeneous problem $T(\xi)$ has a unique solution.

It can be verified by direct calculation that $F(\lambda, \kappa, \xi, \tau)$ is an analytic function of $\lambda \in \Lambda(\xi)$ and a continuous function of $\tau \in [0, \tau_0]$ in the strong operator norm (where τ_0 is an arbitrary positive number). By virtue of this fact, and since $F(\lambda, \kappa, \xi, \tau)$ is a compact operator in H, it satisfies the conditions of the analytical Fredholm theorem [17] and the following proposition is true.

Theorem 4.2. The quasi-eigenfrequencies of problem $T(\xi)$ are discrete on the Riemann surface $\Lambda(\xi)$ and depend continuously on $\tau \in [0, \tau_0]$ and $\xi \in \Xi^{(-)} \cup \Xi^{(+)}$.

Note that this theorem does not state that quasi-eigenvalues exist—that question requires a special investigation.

If $\tau \to 0$ $(\tau \to +\infty)$, problem $T(\xi)$ splits into two problems. In the domain Ω_1, the limiting problem describes waves propagating along a hard (soft) surface, while in Ω_2 they describe waves propagating along a soft (hard) surface. By Theorem 4.2, the waveguide and anomalous properties of problem T for small (large) τ are determined by the waveguide and anomalous properties of problems N and D in the appropriate domains.

Waveguide and anomalous properties: the fine structure of the spectrum. The various propositions proved above imply the following.

Theorem 4.3. If $0 < \tau \ll 1$ (or $1 \ll \tau$), then a one-dimensionally periodic interface possesses the waveguide property. In the interval $(0, \pi/\varkappa)$ (or $(0, \pi)$) the waveguide frequencies of problem T are close to the corresponding waveguide frequencies of problem N. If G is a normally illuminated surface, then the waveguide functions and frequencies of problem T are close to the corresponding waveguide functions and frequencies of problem N.

Remark 4.2. (The whispering interface effect.) By Theorem 4.3, any interface with $0 < \tau \ll 1$ (or $1 \ll \tau$) has a passband in the low-frequency domain.

In the case of an interface of the second type, anomalous oscillations may occur. If Green's function (2.1) is modified so as to satisfy identities (3.6), the proof of the next proposition is analogous to that of the previous theorems.

Theorem 4.4. If $0 < \tau \ll 1$ (or $1 \ll \tau$), then a one-dimensionally periodic interface of the second type possesses the anomalous property. In the interval $(\pi/\varkappa, 2\pi/\varkappa)$ (or $(\pi, 2\pi)$) the anomalous frequencies of problem T are close to the corresponding anomalous frequencies of problem N.

The theorems proved above enable us to state that the fine structure of the spectrum of problem T for $0 < \tau \ll 1$ (or $1 \ll \tau$) does not differ in the frequency band $(0, 2\pi/\varkappa)$ (or $(0, 2\pi)$) from the structure of the spectrum of problem N shown in Fig. 6.

5. CONCLUSION

Let λ_* be a waveguide frequency. It follows from (1.9) that the waveguide function decreases as the distance from the surfaces increases as $\exp(-|x|\sqrt{(\xi^2-\lambda_*^2)})$. This means that it is exponentially localized about the surface. The spatial period of an elementary waveguide packet has the form $Y = 2\pi/\xi$, it varies over the interval $(2, \infty)$ and is unrelated to the period of the surface. The pass and stop bands do not depend on the direction in which the waveguide mode is propagating. The mechanics of wave propagation described by waveguide functions along a periodic surface are known [20] and agree with the mechanical analogues—oscillator trains. The mechanical analogue of anomalous oscillations are in-phase oscillations of a sequence of coupled mathematical pendulums.

The influence of the third dimension on the waveguide and anomalous properties of a OPS may be verified by direct calculation [12, 16]. The variables separate, and the investigation of waveguide and anomalous properties of a three-dimensional structure reduces to solving the two-dimensional problems described above. If λ_* is a waveguide or anomalous frequency of a two-dimensional problem and the dependence on the third coordinate has the form $\exp(ikz)$, then $\lambda_k = \sqrt{(\lambda_*^2 + (k))^2}$ is the corresponding waveguide or anomalous frequency of the three-dimensional problem. For a waveguide mode, the wave vector in this case is (ξ, k). By Theorem 2.1, we can state that any hard surface of the type described has non-trivial waveguide modes for every fixed wave vector.

The investigations described in this paper enable one to describe resonance phenomena about the surface in the case when the source of the oscillations is a periodic function of time. In the problems investigated above, there are two types of resonance phenomena.

Resonance of a spatially localized type is due to the fact that for some waveguide frequencies the group velocity of propagation of the waveguide packet is zero [16]. Hence the energy of the source will be localized in its neighbourhood.

Resonance of the synchrocyclotron type is observed when the spatial periodicity of the chain of sources and the interface are identical, the waveguide number is equal to the phase shift of the oscillations of the sources in different fundamental domains of the translation group, and the waveguide frequency is identical with the frequency of the sources. In that case, the amplitude of the travelling waveguide packet increases.

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